How to Avert an Identity Crisis in Math

The term *identity* is used in several different ways in mathematics:

1. Identity Equation: An equation that is tautologically true.
2. Identity Element: An element that when combined with any other element under a certain operation, leaves that element unchanged. Examples include 0 under real number addition, 1 under real number multiplication, or the identity matrix I under matrix multiplication.
3. Identity Function: A function that maps any element from the domain to itself, such as $f(x) = x$.

It will be clear from context which of these uses we intend when we simply use the term 'identity.' We will focus specifically on identity equations here.

What Exactly is an Identity Equation?

- An identity equation must be an equation. In order to be considered an identity, a statement must contain two expressions that are related by an equals sign (=).
- Secondly, an identity must be tautologically true (true under every interpretation). Another way of saying this is that an identity equation is true for every possible assignment of values to the variable(s) it contains.

Examples:

1. $(x + y)^2 = x^2 + 2xy + y^2$
   is an identity since it is an equation that holds true regardless of what values we pick for the variables x and y.
2. $x + 2 = 10$
   is **not** an identity since it is false whenever x is assigned any numerical value other than 8.
3. $x + 1 \geq x$
   is a tautology since it is true for all values of x, but is **not** an identity since it is not an equation. It does not assert that two expressions are identical.

How to Prove an Identity
There are two ways to prove an identity:

1. Start with one side of the equation and transform it into the other side in a single chain of equalities. It is usually easier to begin with the more complex side and simplify to obtain the other side.

   Example: Prove the identity $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$.

   We begin with the right-hand side since it is the more complex one. We have

   $$(a - b)(a^2 + ab + b^2) = a^3 + a^2b + ab^2 - a^2b - ab^2 - b^3 = a^3 - b^3.$$

   We have obtained the other side of the equation we are trying to prove, so this proves the identity.

2. Start with an equation that is known to be true, and manipulate both sides until you obtain the equation that you are trying to prove.

   Example: Prove the identity $\cos^2\theta = (1 - \sin\theta)(1 + \sin\theta)$.

   We know $\sin^2\theta + \cos^2\theta = 1$ to be true, so we begin by writing that equation. Then we have

   $$\begin{array}{rrcl} & \sin^2\theta + \cos^2\theta & = & 1 \\ \Rightarrow & \cos^2\theta & = & 1 - \sin^2\theta \\ \Rightarrow & \cos^2\theta & = & (1 - \sin\theta)(1 + \sin\theta) \end{array}$$

Rembember, in a mathematical proof, every step you write must be both true and justified.

How NOT to Prove an Identity
Be sure to avoid the following common errors when proving an identity.

1. Don't assume the identity to prove the identity. This means don't begin by writing the very equation you are trying to prove true.

2. Don't look only at specific values. Identities must hold true for **all** possible values the variable(s) can take on. Showing that an equation holds for *some* values of the variable(s) does nothing to prove it is true in general.